

ON THE STABILITY OF THE EXISTENCE OF FIXED POINTS FOR THE PROJECTION-ITERATIVE METHODS WITH RELAXATION

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ABSTRACT. We consider an α -relaxed projection $P_A^\alpha : H \rightarrow H$ given by $P_A^\alpha(x) = \alpha P_A(x) + (1 - \alpha)x$ where $\alpha \in [0, 1]$ and P_A is the projection onto a non-empty, convex and closed subset A of the real Hilbert space H . We characterise all the sets $F \subset [0, 1]$ such that for some non-empty, convex and closed subsets $A_1, A_2, \dots, A_k \subset H$ the composition $P_{A_k}^\alpha P_{A_{k-1}}^\alpha \dots P_{A_1}^\alpha$ has a fixed point iff $\alpha \in F$. It proves, that if $\dim H \geq 3$ and $k \geq 3$ then the class of the described above sets F of coefficients α is exactly the class of F_σ subsets of $[0, 1]$ containing 0.

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The theory of fixed points plays a great role in applications. In particular, researchers investigated fixed points of compositions $P_{A_k} P_{A_{k-1}} \dots P_{A_1}$ of projections onto non-empty convex subsets A_1, \dots, A_k of the real Hilbert (or Euclidean) space. For example Bregman ([1]) finds points in the intersection $A_1 \cap \dots \cap A_k$ using cyclic iterations of the form $x_{n+1} = P_{A_{i_n}} x_n$, where (i_n) is the cyclic sequence $(1, 2, \dots, k, 1, 2, \dots, k, \dots)$. Bregman provides conditions which assure that the sequence (x_n) converges (even in the case of other linear metric spaces). The problem of the convergence of the iterative methods of this type is closely related to the existence of fixed points. If $\dim H < \infty$ then the convergence of (x_n) for every starting point is equivalent to the existence of a common fixed point of the projections P_{A_i} .

Moreover, if $\dim H < \infty$ and $P_{A_k} P_{A_{k-1}} \dots P_{A_1}$ has a fixed point then for every $x \in H$ the sequence $((P_{A_k} P_{A_{k-1}} \dots P_{A_1})^n x)$ is convergent. However, if $\dim H = \infty$ then the existence of a fixed point of the composition $P_{A_k} P_{A_{k-1}} \dots P_{A_1}$ does not imply the norm convergence of $((P_{A_k} P_{A_{k-1}} \dots P_{A_1})^n x)$ for every $x \in H$, even if $A_1 \cap \dots \cap A_k \neq \emptyset$ and $k = 2$ (cf. remarkable examples in [4] and [5]). Despite these negative results the investigation of fixed points of compositions $P_k P_{k-1} \dots P_1$ is the natural first step in research of iterations $((P_k P_{k-1} \dots P_1)^n x)$, where P_1, \dots, P_k are generalisations of the projections P_{A_1}, \dots, P_{A_k} .

One of possible generalisations of projections arise if we consider the relaxation parameter which is commonly used in the iterative methods to control the rate of the convergence and the regularity of trajectories. In the case of projections introducing the relaxation parameter α replaces the projection $P_A x = x + (P_A x - x)$ with a map $x + \alpha(P_A x - x) = \alpha P_A(x) + (1 - \alpha)x$. This leads to the following definition:

Definition 1. Let H be a real Hilbert space, let $A \subset H$ be a non-empty, convex and closed subset of H and let $\alpha \in [0, 1]$. An α -relaxed projection (or α -projection) onto A is the function $P_A^\alpha : H \rightarrow H$ given

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by

$$P_A^\alpha(x) = \alpha P_A(x) + (1 - \alpha)x,$$

where by $P_A : H \rightarrow A$ we denote the projection onto A .

In the paper we concentrate on α -projections but other generalisations of projections had also been used. For example De Pierro ([3]) considered iterations of convex combinations of projections.

Recently, De Pierro and Cegielski (oral communication, [2]) formulated the following interesting problem concerning fixed points: Let A_1, A_2, A_3 be non-empty, convex and closed subsets of the Hilbert space H and let $\alpha \in (0, 1)$. Is the existence of a fixed point of the composition $P_{A_3}P_{A_2}P_{A_1}$ equivalent to the existence of a fixed point of the composition $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\alpha$? The answer is negative. Moreover, we have the following general result:

Theorem 1. *Let H be a Hilbert space, $\dim H \geq 3$, let $k \geq 3$ be an integer and let $F \subset [0, 1]$. The following conditions are equivalent:*

- (i) *There exist non-empty, convex and closed subsets $A_1, A_2, \dots, A_k \subset H$ satisfying*

$$F = \{\alpha \in [0, 1] : P_{A_k}^\alpha P_{A_{k-1}}^\alpha \dots P_{A_1}^\alpha \text{ has a fixed point}\},$$

- (ii) *$0 \in F$ and F is an F_σ subset of $[0, 1]$.*

It can be shown that if $\dim H = 1$ or $k = 1$ then the only set F satisfying (i) is $[0, 1]$. If $k = 2$ and $\dim H \geq 2$ then two sets F satisfy (i), namely $\{0\}$ and $[0, 1]$. It proves, that if $H = \mathbb{R}^2$ and $k \geq 3$ then the class of sets F satisfying (i) depends on k and its full characterization is still an open problem.

Theorem 1 is an immediate consequence of the following two propositions

Proposition 1. *If F is an F_σ subset of $[0, 1]$, $0 \in F$ and $k \geq 3$ then*

$$F = \{\alpha \in [0, 1] : P_{A_k}^\alpha P_{A_{k-1}}^\alpha \dots P_{A_1}^\alpha \text{ has a fixed point}\},$$

for some non-empty, convex and closed subsets $A_1, A_2, \dots, A_k \subset \mathbb{R}^3$.

Proposition 2. *If A_1, A_2, \dots, A_k are non-empty, convex and closed subsets of a Hilbert space H then for every $r > 0$ the set*

$$F_r = \{\alpha \in [0, 1] : P_{A_k}^\alpha P_{A_{k-1}}^\alpha \dots P_{A_1}^\alpha \text{ has a fixed point } x \text{ satisfying } \|x\| \leq r\}$$

is closed in $[0, 1]$.

Proof of Theorem 1. To show that (1) implies (2) it is enough to observe that $P_{A_k}^0 P_{A_{k-1}}^0 \dots P_{A_1}^0$ is the identity (hence $0 \in F$) and that $F = \bigcup_{r \in \mathbb{N}} F_r$, where F_r 's are the closed sets defined in Proposition 2.

Now, let F be any F_σ subset of $[0, 1]$, $0 \in F$ and let $k \geq 3$. By Proposition 1 we have

$$F = \{\alpha \in [0, 1] : P_{A_k}^\alpha P_{A_{k-1}}^\alpha \dots P_{A_1}^\alpha \text{ has a fixed point}\},$$

for some non-empty, convex and closed subsets $A_1, A_2, \dots, A_k \subset \mathbb{R}^3$. Using any isometric embedding of \mathbb{R}^3 into H we obtain that (2) implies (1). \square

1. PROOF OF PROPOSITION 1

Proposition 1 is a consequence of the following lemma

Lemma 1. *If F is an F_σ subset of $[0, 1]$ and $0 \in F$ then for some non-empty, convex and closed sets $A_1, A_2, A_3 \subset \mathbb{R}^3$ one has:*

- (i) *if $\alpha \in F$ and $\beta \in [0, 1]$ then $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$ has a fixed point,*
- (ii) *if $\alpha \in [0, 1] \setminus F$ and $\beta \in (0, 1]$ then $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$ has no fixed point.*

Proof of Proposition 1. Let F be an F_σ subset of $[0, 1]$ and let A_1, A_2 and A_3 be given by Lemma 1. Then $\alpha \in F$ iff $P_{A_3}^\alpha P_{A_2}^\alpha (P_{A_1}^\alpha)^{k-2} = P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^{1-(1-\alpha)^{k-2}}$ has a fixed point (we put $\beta = 1 - (1-\alpha)^{k-2}$). \square

The sets A_1, A_2 and A_3 demanded in Lemma 1 will be defined as $A_1 = \{(x, y, z) : z \geq 0, y \geq f(x, z)\}$, $A_2 = \{(1, 0, z) : z \geq 0\}$, $A_3 = \{(0, 0, z) : z \geq 0\}$ for some continuous convex function $f : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$. The construction of the function f will be the main part of the proof. In particular we will use an auxiliary function φ defined by the following lemma.

Lemma 2. *Let $B_1 = \{(x, y) : y \geq x^2\}$, $B_2 = \{(1, 0)\}$, $B_3 = \{(0, 0)\}$ be subsets of \mathbb{R}^2 . Then for every $\alpha, \beta \in (0, 1]$ the composition $P_{B_3}^\alpha P_{B_2}^\alpha P_{B_1}^\beta$ has a unique fixed point $\mathbf{u}_{\alpha, \beta}$. Moreover, there exists a decreasing and continuous function $\varphi : (0, 1] \rightarrow [0, 1]$ such that $P_{B_1}(\mathbf{u}_{\alpha, \beta}) = (\varphi(\alpha), \varphi(\alpha)^2)$ for every $\alpha, \beta \in (0, 1]$.*

Proof. The existence and the uniqueness of the fixed point follows by the Banach fixed point theorem for the contraction $P_{B_3}^\alpha P_{B_2}^\alpha P_{B_1}^\beta$. Let us denote $\mathbf{x}_{\alpha, \beta} = (x_{\alpha, \beta}, x_{\alpha, \beta}^2) = P_{B_1}(\mathbf{u}_{\alpha, \beta})$. Then

$$\begin{aligned} \mathbf{u}_{\alpha, \beta} &= P_{B_3}^\alpha P_{B_2}^\alpha P_{B_1}^\beta(\mathbf{u}_{\alpha, \beta}) = P_{B_3}^\alpha P_{B_2}^\alpha((1-\beta)\mathbf{u}_{\alpha, \beta} + \beta\mathbf{x}_{\alpha, \beta}) \\ &= (1-\alpha)^2(1-\beta) \cdot \mathbf{u}_{\alpha, \beta} + (1-\alpha)^2\beta \cdot \mathbf{x}_{\alpha, \beta} + (1-\alpha)\alpha \cdot (1, 0) + \alpha \cdot (0, 0), \end{aligned}$$

hence

$$(1) \quad \frac{1 - (1-\alpha)^2(1-\beta)}{\alpha} \cdot (\mathbf{u}_{\alpha, \beta} - \mathbf{x}_{\alpha, \beta}) = (\alpha - 2) \cdot \mathbf{x}_{\alpha, \beta} + (1-\alpha) \cdot (1, 0).$$

By $\mathbf{x}_{\alpha, \beta} = P_{B_1}(\mathbf{u}_{\alpha, \beta})$ it follows that $\mathbf{u}_{\alpha, \beta} - \mathbf{x}_{\alpha, \beta}$ is orthogonal to the tangent to B_1 at $\mathbf{x}_{\alpha, \beta}$, hence $(\mathbf{u}_{\alpha, \beta} - \mathbf{x}_{\alpha, \beta}) \perp (1, 2x_{\alpha, \beta})$. From (1) we obtain

$$((\alpha - 2)x_{\alpha, \beta} + (1-\alpha), (\alpha - 2)x_{\alpha, \beta}^2) \cdot (1, 2x_{\alpha, \beta}) = 0,$$

which is equivalent to

$$2x_{\alpha, \beta}^3 + x_{\alpha, \beta} = \frac{1-\alpha}{2-\alpha}.$$

Since the function $\psi(\alpha) = \frac{1-\alpha}{2-\alpha}$ is decreasing and continuous on $(0, 1]$ and the function $\chi(x) = 2x^3 + x$ is increasing and continuous on \mathbb{R} and $\psi((0, 1]) = [0, \frac{1}{2}) \subset \chi([0, 1])$, we obtain that $x_{\alpha, \beta} = \chi^{-1}(\psi(\alpha)) \in [0, 1]$ does not depend on β and it is the decreasing and continuous function of α . We put $\varphi(\alpha) := x_{\alpha, \beta}$. \square

Letting $\varphi(0) = \lim_{\alpha \rightarrow 0} \varphi(\alpha)$ we extend φ to continuous and decreasing $\varphi : [0, 1] \rightarrow [0, 1]$.

We pass to the construction of the function f for a given F_σ subset $F \subset [0, 1]$ satisfying $0 \in F$. We have $F = \bigcup_{n=1}^\infty F_n$ for some closed sets $F_1 \subset F_2 \subset \dots \subset [0, 1]$. Let $E_n = (\mathbb{R} \setminus (-1, 2)) \cup \varphi(F_n)$ for $n = 1, 2, \dots$

(Here the interval $(-1, 2)$ may be replaced by any open and bounded set containing $\varphi([0, 1])$.) The sets E_n are closed, $\inf E_n = -\infty$ and $\sup E_n = \infty$, hence the functions $a_n, b_n : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$a_n(x) = \max(E_n \cap (-\infty, x]) \quad \text{and} \quad b_n(x) = \min(E_n \cap [x, \infty))$$

are well defined. Note, that if $x \in E_n$ then $a_n(x) = b_n(x) = x$. Otherwise, $(a_n(x), b_n(x))$ is the connected component of $\mathbb{R} \setminus E_n$ containing x . We define

$$(2) \quad f(x, z) = x^2 + \sum_{n=1}^{\infty} c_n g_n(x) h_n(z),$$

where

$$g_n(x) = (x - a_n(x))^3 (b_n(x) - x)^3 \quad \text{and} \quad h_n(z) = (n - z)_+^3 = \begin{cases} (n - z)^3 & \text{for } z \in [0, n] \\ 0 & \text{for } z > n \end{cases}$$

and (c_n) is any sequence with positive terms satisfying $\sum_{n=1}^{\infty} \frac{81^2}{6} n^3 c_n < 1$.

Lemma 3. *The function f defined by (2) satisfies the following conditions:*

- (i) $f(x, z) \geq x^2$ for every $x \in \mathbb{R}$ and $z \geq 0$,
- (ii) $f \in C^2$ and f is convex,
- (iii) For every $x \in \mathbb{R}$ one has: $x \in \bigcup_{n=1}^{\infty} E_n \Leftrightarrow \exists z \geq 0 \ f(x, z) = x^2$,
- (iv) For every $x \in \mathbb{R}$ and $z \geq 0$ if $f(x, z) > x^2$ then $\frac{\partial f}{\partial z}(x, z) < 0$.

Proof. We have

$$g'_n(x) = 3(x - a_n(x))^2 (b_n(x) - x)^2 (a_n(x) + b_n(x) - 2x), \quad h'_n(z) = -3((n - z)_+)^2,$$

$$g''_n(x) = 6(x - a_n(x))(b_n(x) - x)[(a_n(x) + b_n(x) - 2x)^2 - (x - a_n(x))(b_n(x) - x)] \quad \text{and} \quad h''_n(z) = 6(n - z)_+.$$

For every $x \in \mathbb{R}$ and $z > 0$ one has

$$|g_n(x)| \leq 3^6, \quad |g'_n(x)| \leq 3^6, \quad |g''_n(x)| \leq 6 \cdot 3^4,$$

$$|h_n(z)| \leq n^3, \quad |h'_n(z)| \leq 3n^2 \quad \text{and} \quad h''_n(z) \leq 6n.$$

It follows, that the series (2) is uniformly convergent. Moreover, if we try to calculate the first and the second order derivatives of f by the formal differentiation of the series (2) term by term then we obtain a uniformly convergent series with continuous terms. It follows that f is well defined and $f \in C^2$. Since $g_n(x), h_n(z) \geq 0$ we get (i).

We will check the convexity of f by showing that the Hessian matrix $H(f)(x, z)$ is positive semidefinite for every $x \in \mathbb{R}$ and $z > 0$.

$$H(f)(x, z) = \begin{pmatrix} 2 + \sum_{n=1}^{\infty} c_n g''_n(x) h_n(z) & \sum_{n=1}^{\infty} c_n g'_n(x) h'_n(z) \\ \sum_{n=1}^{\infty} c_n g'_n(x) h'_n(z) & \sum_{n=1}^{\infty} c_n g_n(x) h''_n(z) \end{pmatrix}.$$

Clearly $\sum_{n=1}^{\infty} c_n g_n(x) h''_n(z) \geq 0$ and

$$2 + \sum_{n=1}^{\infty} c_n g''_n(x) h_n(z) \geq 2 - \sum_{n=1}^{\infty} c_n \cdot 6 \cdot 3^4 \cdot n^3 > 2 - \sum_{n=1}^{\infty} \frac{81^2}{6} n^3 c_n > 1.$$

Moreover,

$$\begin{aligned}
\det H(f)(x, z) &= \left(2 + \sum_{n=1}^{\infty} c_n g_n''(x) h_n(z) \right) \left(\sum_{n=1}^{\infty} c_n g_n(x) h_n''(z) \right) - \left(\sum_{n=1}^{\infty} c_n g_n'(x) h_n'(z) \right)^2 \\
&\geq 1 \cdot \sum_{n=1}^{\infty} c_n g_n(x) h_n''(z) - \left(\sum_{n=1}^{\infty} c_n g_n'(x) h_n'(z) \right)^2 \\
&\geq \sum_{n=1}^{\infty} \frac{81^2}{6} n^3 c_n \cdot \sum_{n=1}^{\infty} 6c_n (x - a_n(x))^3 (b_n(x) - x)^3 (n - z)_+ \\
&\quad - \left(\sum_{n=1}^{\infty} 9c_n (x - a_n(x))^2 (b_n(x) - x)^2 (a_n(x) + b_n(x) - 2x) ((n - z)_+)^2 \right)^2 \\
&\geq \sum_{n=1}^{\infty} \frac{81^2}{6} n^3 c_n \cdot \sum_{n=1}^{\infty} 6c_n (x - a_n(x))^3 (b_n(x) - x)^3 (n - z)_+ \\
&\quad - \left(\sum_{n=1}^{\infty} 81 n^{\frac{3}{2}} c_n (x - a_n(x))^{\frac{3}{2}} (b_n(x) - x)^{\frac{3}{2}} ((n - z)_+)^{\frac{1}{2}} \right)^2 \geq 0
\end{aligned}$$

In the above we used inequalities $0 \leq x - a_n(x) \leq 3$, $0 \leq b_n(x) - x \leq 3$ (hence $|a_n(x) + b_n(x) - 2x| \leq 3$) and, finally, the Schwartz inequality. We obtained that the Hessian matrix $H(f)(x, z)$ is positive semidefinite, hence we have (ii).

Now, we will show (iii). If $x \in \bigcup_{n=1}^{\infty} E_n$ then $x \in E_{n_0}$ for some n_0 and (since $E_1 \subset E_2 \subset \dots$) $x \in E_n$ for every $n \geq n_0$. If $x \in E_n$ then $g_n(x) = 0$ (by the definition of g_n). Consequently, for every $z > n_0$ we have

$$f(x, z) = x^2 + \sum_{n < n_0} c_n g_n(x) \cdot 0 + \sum_{n \geq n_0} c_n \cdot 0 \cdot h_n(z) = x^2.$$

If $x \notin \bigcup_{n=1}^{\infty} E_n$ then $g_n(x) > 0$ for every n . Let $z \geq 0$. Then $z < n_0$ (hence $h_{n_0}(z) > 0$) for some n_0 and we have

$$f(x, z) \geq x^2 + c_{n_0} g_{n_0}(x) h_{n_0}(z) > x^2.$$

Finally, we will show (iv). First observe that for every n , x and z we have $\frac{\partial c_n g_n(x) h_n(z)}{\partial z}(x, z) \leq 0$. It follows that if $f(x, z) > x^2$ then $g_{n_0}(x) > 0$ and $h_{n_0}(z) > 0$ for some n_0 and $\frac{\partial f}{\partial z}(x, z) \leq \frac{\partial c_{n_0} g_{n_0}(x) h_{n_0}(z)}{\partial z}(x, z) < 0$ \square

Proof of Lemma 1. Let F be an F_σ subset of $[0, 1]$ satisfying $0 \in F$. We define the sets $A_1, A_2, A_3 \subset \mathbb{R}^3$ as follows:

$$A_1 = \{(x, y, z) : z \geq 0, y \geq f(x, z)\},$$

$$A_2 = \{(1, 0, z) : z \geq 0\},$$

$$A_3 = \{(0, 0, z) : z \geq 0\},$$

where the continuous convex function $f : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ is defined by (2). Moreover, let $A'_1 = \{(x, y, z) : z \geq 0, y \geq x^2\}$.

If $\alpha = 0$ and $\beta \in [0, 1]$ then $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta = P_{A_1}^\beta$ and every $\mathbf{u} \in A_1$ is a fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$.

Let $\alpha \in F \setminus \{0\}$ and $\beta \in [0, 1]$. Then $\varphi(\alpha) \in \bigcup_{n=1}^{\infty} E_n$ and by Lemma 3 (iii) there exists $z \geq 0$ such that $f(\varphi(\alpha), z) = \varphi(\alpha)^2$, i.e. $(\varphi(\alpha), \varphi(\alpha)^2, z) \in A_1$. Let $(u, v) = \mathbf{u}_{\alpha, \beta}$ be the fixed point of $P_{B_3}^\alpha P_{B_2}^\alpha P_{B_1}^\beta$ given in Lemma 2. Then (u, v, z) is a fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$ (because $A'_1 = B_1 \times [0, \infty)$, $A_2 = B_2 \times [0, \infty)$,

$A_3 = B_3 \times [0, \infty)$ and $z \geq 0$). Moreover, $A_1 \subset A'_1$ (by Lemma 3 (i)) and $P_{A'_1}(u, v, z) = (\varphi(\alpha), \varphi(\alpha)^2, z) \in A_1$. It follows that $P_{A_1}(u, v, z) = P_{A'_1}(u, v, z)$, hence (u, v, z) is a fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$.

Finally, let $\alpha \in [0, 1] \setminus F$ and let $\beta \in (0, 1]$. Assume, aiming at a contradiction, that (u, v, z) is the fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$. Since $\alpha, \beta > 0$, we obtain that (u, v, z) is outside the set A_1 and $z \geq 0$ which means that $(u_1, v_1, z_1) := P_{A_1}(u, v, z)$ satisfies $v_1 = f(u_1, z_1)$.

If $v_1 = f(u_1, z_1) > u_1^2$ then by Lemma 3 (iv) we have $\frac{\partial f}{\partial z}(u_1, z_1) < 0$. Consequently $z_1 > z$. It follows that the last coordinate of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta(u, v, z)$ which is equal to the last coordinate of $P_{A_1}^\beta(u, v, z) = \beta z_1 + (1 - \beta)z$ is greater than z . We obtained a contradiction, since (u, v, z) is a fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$.

Thus $v_1 = f(u_1, z_1) = u_1^2$, hence (u_1, v_1, z_1) is located at the boundaries of both A_1 and A'_1 . Since for both A_1 and A'_1 there exist tangent planes at (u_1, v_1, z_1) and $A_1 \subset A'_1$, it follows that these two planes are equal. Consequently, $P_{A'_1}(u, v, z) = P_{A_1}(u, v, z) = (u_1, v_1, z_1)$, hence (u, v, z) is a fixed point of $P_{A_3}^\alpha P_{A_2}^\alpha P_{A'_1}^\beta$. By Lemma 2 we obtain $(u_1, v_1, z_1) = (u_1, f(u_1, z_1), z_1) = (\varphi(\alpha), \varphi(\alpha)^2, z_1)$, in particular $f(\varphi(\alpha), z_1) = \varphi(\alpha)^2$. Finally, by Lemma 3 (iii) we get $\varphi(\alpha) \in \bigcup_{n=1}^\infty E_n$, hence $\alpha \in F$. We got the contradiction. Hence $P_{A_3}^\alpha P_{A_2}^\alpha P_{A_1}^\beta$ has no fixed point, as required. \square

2. PROOF OF PROPOSITION 2

Let $P^\alpha := P_{A_k}^\alpha P_{A_{k-1}}^\alpha \cdots P_{A_1}^\alpha$.

If $\dim H < \infty$ then Proposition 2 is a consequence of the compactness of the closed balls in H . Indeed, let $\alpha_1, \alpha_2, \dots \in F_r$ and let $\alpha_n \rightarrow \alpha_0$. Then for every n we have $P^{\alpha_n} x_n = x_n$ for some $x_n \in H$ satisfying $\|x_n\| \leq r$. Considering subsequences of (x_n) and (α_n) we may assume that (x_n) is convergent to some $x_0 \in H$ with $\|x_0\| \leq r$. Using the continuity of the function $(\alpha, x) \mapsto P^\alpha x$ we obtain

$$P^{\alpha_0} x_0 = \lim_{n \rightarrow \infty} P^{\alpha_n} x_n = \lim_{n \rightarrow \infty} x_n = x_0.$$

It follows that $\alpha_0 \in F_r$ hence F_r is closed.

If $\dim H = \infty$ then the ball in H is not compact and the above reasoning does not work. One idea is to consider the weak topology on H (instead of the norm topology). Unfortunately it still does not work, because the projection onto a closed convex set in H does not need to be weakly continuous. For these reasons if $\dim H = \infty$ then the proof is more complicated. The idea is as follows: Using the compactness of a closed ball in the weak topology we will find x_0 which is a condensation point in the weak topology of the defined above sequence (x_n) and then we will construct a sequence (u_M) satisfying $\|u_M - x_0\| \rightarrow 0$ and $\|P^{\alpha_0}(u_M) - u_M\| \rightarrow 0$ for $M \rightarrow \infty$. Then, by the continuity of $x \mapsto P^{\alpha_0}x$ in the norm topology we obtain $P^{\alpha_0}(x_0) = x_0$.

Lemma 4. *Let $M \in \mathbb{N}$ and let $(y_i^n)_{i=1, \dots, M}^{n \in \mathbb{N}}$ and $(y^n)_{n \in \mathbb{N}}$ be systems of elements of H satisfying:*

- (i) $\lim_{n \rightarrow \infty} \|y_i^n\| = 1$ for $i = 1, \dots, M$,
- (ii) $\lim_{n \rightarrow \infty} (y_i^n, y_j^n) = 0$ for $i \neq j$, $i, j = 1, \dots, M$,
- (iii) $\limsup_{n \rightarrow \infty} \|y^n - y_i^n\|^2 \leq \frac{M-1}{M}$ for $i = 1, \dots, M$.

Then $\lim_{n \rightarrow \infty} \|y^n - \frac{y_1^n + \dots + y_M^n}{M}\| = 0$.

Proof. We have $y^n = z^n + \sum_{i=1}^M \alpha_i^n y_i^n$ for some $\alpha_i^n \in \mathbb{R}$ and $z^n \in H$ with $z^n \perp y_i^n$ for $i = 1, \dots, M$. Then, by (i), (ii) and (iii), for large enough n one has

$$\begin{aligned} 4 > (\|y^n - y_i^n\| + \|y_i^n\|)^2 &\geq \|y^n\|^2 \geq \left\| \sum_{i=1}^M \alpha_i^n y_i^n \right\|^2 = \sum_{i=1}^M (\alpha_i^n)^2 \|y_i^n\|^2 + \sum_{i \neq j} \alpha_i^n \alpha_j^n (y_i^n, y_j^n) \\ &\geq \sum_{i=1}^M (\alpha_i^n)^2 \|y_i^n\|^2 - \sum_{i \neq j} \frac{(\alpha_i^n)^2 + (\alpha_j^n)^2}{2} |(y_i^n, y_j^n)| = \sum_{i=1}^M (\alpha_i^n)^2 \left(\|y_i^n\|^2 - \sum_{j \neq i} |(y_i^n, y_j^n)| \right) \geq \frac{1}{2} \sum_{i=1}^M (\alpha_i^n)^2. \end{aligned}$$

It follows that all α_i^n 's are bounded. Moreover, for every $l = 1, \dots, M$ one has

$$\|y^n - y_l^n\|^2 = \|z^n\|^2 + \sum_{i \neq l} (\alpha_i^n)^2 \|y_i^n\|^2 + (\alpha_l^n - 1)^2 \|y_l^n\|^2 + \sum_{i \neq j, i, j \neq l} \alpha_i^n \alpha_j^n (y_i^n, y_j^n) + \sum_{i \neq l} \alpha_i^n (\alpha_l^n - 1) (y_i^n, y_l^n),$$

hence (taking limsup in the above and using (i), (ii) and (iii) and the boundedness of α_i^n 's)

$$\limsup_{n \rightarrow \infty} \left(\|z^n\|^2 + \sum_{i=1}^M (\alpha_i^n)^2 - 2\alpha_l^n + 1 \right) \leq \frac{M-1}{M}.$$

Summing the above inequalities with $l = 1, \dots, M$ we obtain

$$\limsup_{n \rightarrow \infty} \left(M\|z^n\|^2 + M \sum_{i=1}^M (\alpha_i^n)^2 - 2 \sum_{l=1}^M \alpha_l^n + M \right) \leq M-1,$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \left(M\|z^n\|^2 + M \sum_{i=1}^M \left(\alpha_i^n - \frac{1}{M} \right)^2 \right) \leq 0.$$

It follows that $\lim_{n \rightarrow \infty} \|z^n\| = 0$ and $\lim_{n \rightarrow \infty} \alpha_i^n = \frac{1}{M}$ for $i = 1, \dots, M$, hence

$$\left\| y_n - \frac{y_1^n + \dots + y_M^n}{M} \right\| \leq \|z_n\| + \sum_{i=1}^M |\alpha_i^n - \frac{1}{M}| \|y_i^n\| \rightarrow 0.$$

□

We are ready to prove Proposition 2 in the general case. Let $\alpha_1, \alpha_2, \dots \in F_r$ and let $\alpha_n \rightarrow \alpha_0$. For every n let $x_n \in H$ satisfy $P^{\alpha_n} x_n = x_n$ and $\|x_n\| \leq r$. Considering subsequences of (x_n) and (α_n) we may assume that (x_n) is weakly convergent to some $x_0 \in H$ with $\|x_0\| \leq r$. Again, considering subsequences of (x_n) and (α_n) we may assume that:

- $\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$,

or

- $\lim_{n \rightarrow \infty} \|x_n - x_0\| = \lambda$ for some $\lambda > 0$ and $(x_n - x_0, x_m - x_0) \rightarrow 0$ when $n, m \rightarrow \infty$.

If $\lim_{n \rightarrow \infty} \|x_n - x_0\| = 0$ then (similarly as in finite dimensional case) by the continuity of the function $(\alpha, x) \mapsto P^\alpha x$ we obtain

$$P^{\alpha_0} x_0 = \lim_{n \rightarrow \infty} P^{\alpha_n} x_n = \lim_{n \rightarrow \infty} x_n = x_0$$

and we are done.

Otherwise (if $\lim_{n \rightarrow \infty} \|x_n - x_0\| = \lambda$ for some $\lambda > 0$ and $(x_n - x_0, x_m - x_0) \rightarrow 0$ when $n, m \rightarrow \infty$) we proceed as follows: For any fixed $M \in \mathbb{N}$ we define

$$\begin{aligned} y_i^n &= \frac{1}{\lambda} (P^{\alpha_0}(x_{n+i}) - x_0) \quad \text{for } n \in \mathbb{N} \text{ and } i = 1, \dots, M, \\ y^n &= \frac{1}{\lambda} \left(P^{\alpha_0} \left(\frac{x_{n+1} + \dots + x_{n+M}}{M} \right) - x_0 \right) \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

We will check that $(y_i^n)_{i=1,\dots,M}^{n \in \mathbb{N}}$ and $(y^n)_{n \in \mathbb{N}}$ satisfy the assumptions of Lemma 4.

(i). We have

$$\|y_i^n\| = \left\| \frac{1}{\lambda} (P^{\alpha_0}(x_{n+i}) - x_0) \right\| = \left\| \frac{x_{n+i} - x_0}{\lambda} + \frac{P^{\alpha_0}(x_{n+i}) - x_{n+i}}{\lambda} \right\|,$$

which yields (i), because $\left\| \frac{x_{n+i} - x_0}{\lambda} \right\| \rightarrow 1$ and $\|P^{\alpha_0}(x_{n+i}) - x_{n+i}\| = \|P^{\alpha_0}(x_{n+i}) - P^{\alpha_{n+i}}(x_{n+i})\| \rightarrow 0$.

Similarly (by $(x_{n+i} - x_0, x_{n+j} - x_0) \rightarrow 0$ for $i \neq j$ and $n \rightarrow \infty$) we obtain (ii).

(iii). We have

$$\begin{aligned} \|y^n - y_i^n\|^2 &= \left\| \frac{P^{\alpha_0} \left(\frac{x_{n+1} + \dots + x_{n+M}}{M} \right) - P^{\alpha_0}(x_{n+i})}{\lambda} \right\|^2 \leq \frac{1}{\lambda^2} \left\| \frac{x_{n+1} + \dots + x_{n+M}}{M} - x_{n+i} \right\|^2 \\ &= \frac{1}{\lambda^2} \left\| \sum_{j \neq i} \frac{1}{M} (x_{n+j} - x_0) - \frac{M-1}{M} (x_{n+i} - x_0) \right\|^2 \end{aligned}$$

and by $\|x_{n+i} - x_0\| \rightarrow \lambda$ and $(x_{n+i} - x_0, x_{n+j} - x_0) \rightarrow 0$ for $i \neq j$ and $n \rightarrow \infty$ we obtain

$$\limsup_{n \rightarrow \infty} \|y^n - y_i^n\|^2 \leq \frac{1}{\lambda^2} \left(\sum_{j \neq i} \frac{1}{M^2} \lambda^2 + \frac{(M-1)^2}{M^2} \lambda^2 \right) = \frac{M-1}{M}.$$

By Lemma 4 we obtain

$$\lim_{n \rightarrow \infty} \left\| P^{\alpha_0} \left(\frac{x_{n+1} + \dots + x_{n+M}}{M} \right) - \frac{P^{\alpha_0}(x_{n+1}) + \dots + P^{\alpha_0}(x_{n+M})}{M} \right\| = 0$$

which (by $\|P^{\alpha_0}(x_{n+i}) - x_{n+i}\| \rightarrow 0$) is equivalent to

$$(3) \quad \lim_{n \rightarrow \infty} \left\| P^{\alpha_0} \left(\frac{x_{n+1} + \dots + x_{n+M}}{M} \right) - \frac{x_{n+1} + \dots + x_{n+M}}{M} \right\| = 0.$$

On the other hand, for large enough n we have

$$\left\| \frac{x_{n+1} + \dots + x_{n+M}}{M} - x_0 \right\|^2 = \frac{1}{M^2} \left(\sum_{i=1}^M \|x_{n+i} - x_0\|^2 + \sum_{i \neq j} (x_{n+i} - x_0, x_{n+j} - x_0) \right) < \frac{2\lambda^2}{M}.$$

By the above inequality and by (2) it follows that choosing large enough n and letting

$$u_M := \frac{x_{n+1} + \dots + x_{n+M}}{M}$$

we have $\|P^{\alpha_0}(u_M) - u_M\| < \frac{1}{M}$ and $\|u_M - x_0\| < \lambda \sqrt{\frac{2}{M}}$.

We constructed the sequence (u_M) satisfying $\|u_M - x_0\| \rightarrow 0$ and $\|P^{\alpha_0}(u_M) - u_M\| \rightarrow 0$ for $M \rightarrow \infty$.

Hence $P^{\alpha_0}(x_0) = \lim_{M \rightarrow \infty} P^{\alpha_0}(u_M) = \lim_{M \rightarrow \infty} u_M = x_0$. Thus $\alpha_0 \in F_r$ and F_r is closed.

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